$U(2)$ projectors and 't Hooft-Polyakov monopoles on a fuzzy sphere

P. Valtancoli

Dipartimento di Fisica, Polo Scientifico Universitá di Firenze
and INFN, Sezione di Firenze (Italy)
Via G. Sansone 1, 50019 Sesto Fiorentino, Italy

Abstract

We show how to generalize our method, based on projective modules and matrix models, which enabled us to derive noncommutative monopoles on a fuzzy sphere, to the non-abelian case, recovering known results in literature. We then discuss a possible candidate for deforming the commutative Chern class to the non-commutative case.

1   Introduction

Recently the study of topologically nontrivial configurations [1]-[2]-[3] on a noncommutative manifold [4] has received growing interest. In this framework we have set up a general method with which all the noncommutative topological configurations for a $U(1)$ gauge group could be studied exhaustively [5].

Basic elements of such a method are a combination of mathematical and physical concepts. To identify easily the nontrivial topological configurations we have made use of the concept of projector [6]-[7]-[8], obtaining their classification on a fuzzy sphere for a $U(1)$ gauge group [9].

With a subsequent study we have introduced a natural way for defining noncommutative actions on a fuzzy sphere [10]-[11]-[12]-[13]-[14]-[15], by using the matrix model approach [16]-[17]-[18]-[19]-[20]-[21]-[22]. Finally with a simple link between projectors and matrix models we have associated to the noncommutative projectors the corresponding noncommutative connections [5]-[23]-[24], generalizing Dirac monopoles on a fuzzy sphere [5]-[25].

To demonstrate that our method is quite general, we have studied in this work the case of a nonabelian gauge group, and , as we will show next, we are able to obtain the same results known in literature [25].

Starting from a noncommutative projector with entries belonging to $U(2)$, we show that it is possible to reconstruct, along the same lines of the abelian case, a solution of the matrix model equations of motion, that coincides, in the classical limit, with the 't Hooft-Polyakov monopoles for $SU(2)$ gauge group.

In the final part of this work we discuss a possible candidate for deforming the Chern class and its corresponding topological number. The action we propose is the $3d$ Chern Simons action, which enjoys the property of invariance under deformed diffeomorphisms [26]. While the nonabelian case works fine without any problem, the abelian one appears more obscure and requires a deeper analysis of the commutative limit, to reach smoothness with the Chern class.

2   Review of abelian noncommutative monopoles

The aim of this work is to describe how abelian and non-abelian monopoles can be deformed on a non-commutative sphere. Our research tries to cover both mathematical and physical issues, filling the gap between the projective module point of view and the matrix model formalism, which incorporates the natural definition of a non-commutative gauge theory on

a non-commutative manifold.

Since the abelian case has been already worked out in ref. [5], we briefly remind the principal features in this section so that the reader can easily identify common features and differences with the corresponding non-abelian case.

The starting point is the definition of the fuzzy sphere algebra

$$[\hat{x}_i, \hat{x}_j] = i\alpha\epsilon_{ijk}\hat{x}_k \qquad \sum_i (\hat{x}_i)^2 = R^2$$

$$\alpha = \frac{2R}{\sqrt{N(N+2)}} \tag{2.1}$$

which produces an useful finite truncation of the infinite-dimensional function space of the classical sphere.

Such an algebra can be simply obtained by quantizing the Hopf fibration $S^3 \to S^2$, which is defined in terms of two complex coordinates, constrained to the S^3 sphere :

$$
\begin{aligned}
x_1 &= z_0\bar{z}_1 + z_1\bar{z}_0 \\
x_2 &= i(z_0\bar{z}_1 - z_1\bar{z}_0) \\
x_3 &= |z_0|^2 - |z_1|^2 \qquad |z_0|^2 + |z_1|^2 = 1
\end{aligned}
\tag{2.2}
$$

Substituting into this mapping (2.2) the complex coordinates with oscillator operators a_i satisfying the algebra

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{2.3}$$

the corresponding real combinations are representations of the fuzzy sphere algebra :

$$
\begin{aligned}
\hat{x}_1 &= \frac{\hat{\alpha}}{2}(a_0 a_1^\dagger + a_1 a_0^\dagger) \\
\hat{x}_2 &= i\frac{\hat{\alpha}}{2}(a_0 a_1^\dagger - a_1 a_0^\dagger) \\
\hat{x}_3 &= \frac{\hat{\alpha}}{2}(a_0 a_0^\dagger - a_1 a_1^\dagger) \\
\hat{N} &= a_0^\dagger a_0 + a_1^\dagger a_1
\end{aligned}
\tag{2.4}
$$

The $\hat{\alpha}$ operator has to be defined yet. Restricting the action of the oscillators on representations with fixed total number $\hat{N} = N$, the $\hat{\alpha}$ operator can be taken as the α constant

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$$\hat{\alpha} \to \alpha = \frac{2R}{\sqrt{N(N+2)}} \qquad (2.5)$$

This relation is particularly useful to generalize the monopole projectors at a non-commutative level. Let us recall the classical $k = 1$ case, i.e. the simplest monopole configuration, which is defined by the following projector P_1, function of the complex coordinates z_i:

$$
\begin{aligned}
P_1 &= |\psi_1 >< \psi_1| \\
|\psi_1 > &= \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}
\end{aligned}
\qquad (2.6)
$$

The normalization condition of this vector

$$< \psi_1|\psi_1 >= |z_0|^2 + |z_1|^2 = 1 \qquad (2.7)$$

is satisfied, since the complex coordinates belong to S^3, as in formula (2.2).

In our first paper [9] we noticed that the natural non-commutative extension, based on quantizing the Hopf fibration (eq. (2.3))

$$
\begin{aligned}
P_1 &= |\psi_1 >< \psi_1| \\
|\psi_1 > &= N_1 \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}
\end{aligned}
\qquad (2.8)
$$

really works by choosing the normalization factor as

$$
\begin{aligned}
< \psi_1|\psi_1 > &= 1 \\
N_1 = N_1(\hat{N}) &= \frac{1}{\sqrt{\hat{N}+1}}
\end{aligned}
\qquad (2.9)
$$

Since the fuzzy sphere algebra is a finite type non-commutative geometry, the trace of the non-commutative projector is an integer number :

$$TrP_1 = Tr|\psi_1 >< \psi_1| = N + 2 < Tr1_P = 2(N+1) \qquad (2.10)$$

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It is possible to repeat the same procedure for monopoles with negative charge. For example the projector corresponding to $k = -1$ is defined as :

$$
\begin{aligned}
P_{-1} &= |\psi_{-1} >< \psi_{-1}| \\
|\psi_{-1} > &= N_{-1} \begin{pmatrix} a_0^\dagger \\ a_1^\dagger \end{pmatrix}
\end{aligned}
\tag{2.11}
$$

where the normalization factor is determined by the condition

$$
< \psi_{-1}|\psi_{-1} >= 1 \quad \Rightarrow N_{-1} = N_{-1}(\hat{N}) = \frac{1}{\sqrt{\hat{N}+1}}
\tag{2.12}
$$

Again the trace of P_{-1} is an integer number:

$$
TrP_{-1} = N < Tr1_P = 2(N+1)
\tag{2.13}
$$

Proving that these analogues of the topological excitations of Yang-Mills theory on a sphere satisfy the non-commutative equations of motion requires the definition of a Yang-Mills theory on a fuzzy sphere.

This proof has been firstly achieved in ref. [5] with the introduction of a matrix model $X_i(i = 1, .., 3)$, whose equations of motion contain the fuzzy sphere algebra as particular solutions:

$$
S(\lambda) = S_0 + \lambda S_1 = -\frac{1}{g^2} Tr[\frac{1}{4}[X_i, X_j][X_i, X_j] - \frac{2}{3} i \lambda \alpha \epsilon^{ijk} X_i X_j X_k + \alpha^2(1-\lambda) X_i X_i]
\tag{2.14}
$$

Both actions S_0 and S_1 separately contain the fuzzy sphere algebra as a particular solution, and λ is a generic parameter.

The usual gauge connection is obtained by separating the generic matrix X_i from the background L_i (in the following we will change the variable X_i into αX_i) :

$$
X_i = L_i + A_i \quad [L_i, .] \to_{\alpha \to 0} -i k_i^a \partial_a
\tag{2.15}
$$

The background L_i is equal to the classical Lie derivative on the sphere in the $\alpha \to 0$ (classical limit). Instead the fluctuation A_i is a linear combination of the Yang-Mills connection $A_a(\Omega)$ and an auxiliary scalar field (in the adjoint representation) ϕ:

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$$A_i(\Omega) = k_i^a A_a(\Omega) + \frac{x_i}{R}\phi(\Omega) \quad \Omega = (\theta, \phi) \tag{2.16}$$

A gauge covariant field strength can be defined as

$$
\begin{aligned}
F_{ij} &= [X_i, X_j] - i\epsilon_{ijk}X_k = \\
&= [L_i, A_j] - [L_j, A_i] + [A_i, A_j] - i\epsilon_{ijk}A_k
\end{aligned} \tag{2.17}
$$

In the classical limit it is useful to calculate it in terms of the component fields

$$F_{ij}(\Omega) = k_i^a k_j^b F_{ab} + \frac{i}{R}\epsilon_{ijk}x_k\phi - \frac{x_i}{R}k_j^a D_a\phi + \frac{x_j}{R}k_i^a D_a\phi \tag{2.18}$$

where

$$
\begin{aligned}
F_{ab} &= -i(\partial_a A_b - \partial_b A_a) + [A_a, A_b] \\
D_a\phi &= -i\partial_a\phi + [A_a, \phi]
\end{aligned} \tag{2.19}
$$

By inserting these formulas into the action $S(\lambda)$, its value in the classical limit is the general action:

$$
\begin{aligned}
S(\lambda) &= -\frac{1}{4g_{YM}^2}\int d\Omega[(F_{ab} + (4-2\lambda)i\epsilon_{ab}\phi\sqrt{g})(F^{ab} + (4-2\lambda)i\epsilon^{ab}\frac{\phi}{\sqrt{g}}) + \\
&+ 2g^{ab}D_a\phi D_b\phi + 8(\lambda-2)(\lambda-\frac{3}{2})\phi^2].
\end{aligned} \tag{2.20}
$$

We note that the auxiliary scalar field ϕ can be decoupled from the pure Yang-Mills theory only in the abelian $U(1)$ case and for the special case $\lambda = 2$:

$$
\begin{aligned}
S(2) &= -\frac{1}{4g_{YM}^2}\int d\Omega(F_{ab}F^{ab} - 2\partial_a\phi\partial^a\phi) \\
d\Omega &= \sqrt{g}d\theta d\phi = sin\theta d\theta d\phi \qquad F^{ab} = g^{aa'}g^{bb'}F_{a'b'}.
\end{aligned} \tag{2.21}
$$

When searching a connection between projectors and the matrix model variable X_i, we don't have many choices since the background matrix L_i is the only possible definition of derivative, and we must play with the vector $|\psi>$. The natural guess is

$$X_i = <\psi|L_i|\psi> = L_i + <\psi|[L_i, |\psi>] \tag{2.22}$$

which implies the following representation for the fluctuation field A_i :

$$A_i = <\psi|[L_i, |\psi>] \tag{2.23}$$

Since $[L_i, .] \to_{\alpha \to 0} -ik_i^a \partial_a$, we obtain the well-known classical formula for the monopole connection:

$$A_i \to -ik_i^a <\psi|\partial_a|\psi> \tag{2.24}$$

There is however a problem that may ruin the classical limit. If we use the vectors $|\psi_{\pm 1}>$, depending on the oscillator algebra, rather than the fuzzy sphere algebra, the action of L_i on $|\psi_{\pm 1}>$ is discontinuous in the classical limit. The only way out is redefining the vectors $|\psi_{\pm 1}>$ with an operator acting on the right such that the new vector can be restricted to a fixed total oscillator number $\hat{N} = N$.

It turns out that the correction is only possible for $|\psi_{-1}>$ using a quasi-unitary operator :

$$|\psi_{-1}> \to |\psi'_{-1}> = |\psi_{-1}> U \quad UU^\dagger = 1 \tag{2.25}$$

The quasi-unitary condition keeps the projector P_{-1} invariant. A possible choice turns out to be :

$$U_1 = \sum_{n_1, n_2 = 0}^{\infty} |n_1, n_2><n_1 + 1, n_2| \tag{2.26}$$

It is not difficult to show that the combination X_i is proportional to a reducible representation of the Lie algebra :

$$\begin{aligned}
X_i &= <\psi'_{-1}|L_i|\psi'_{-1}> = \frac{N+2}{N+1} U^\dagger L_i U \\
F_{ij} &= \frac{N+2}{(N+1)^2} i\epsilon_{ijk} U^\dagger L_k U
\end{aligned} \tag{2.27}$$

that is indeed a solution of the non-commutative equations of motion for :

$$\lambda = 2 + \frac{1}{N+1} \tag{2.28}$$

This is a direct deformation of the classical monopole solution. However one can also choose to redefine X_i as

$$X_i = U^\dagger L_i U \quad F_{ij} = 0 \quad \lambda = 2 \tag{2.29}$$

remembering that in this case the classical limit contains not only the monopole field but also a constant scalar field, that due to the $U(1)$ property, remains totally decoupled. In the following we will shift from one to the other formulation indifferently.

3 $U(2)$ projectors and 't Hooft-Polyakov monopoles

In the case of $U(1)$ projectors, our strategy was to extend the well-known classical case, studied in detail in ref. [8].

When generalizing to the non-abelian 't Hooft-Polyakov monopoles, the presence of a non-trivial Higgs field complicates the classical analysis, and it is more convenient starting directly from the matrix model formalism, which simplifies the whole picture.

We therefore prefer to postulate some non-commutative $U(2)$ projectors, whose form is dictated by internal consistency, and then we connect them to known solutions (see ref. [25]), leading to non-commutative extensions of 't Hooft-Polyakov monopoles. The form of $U(2)$ projectors, given in terms of oscillators, can be guessed from the 4d case, where our guide was the classical case of $SU(2)$ BPST instantons, discussed in ref. [24]. This analogy suggests us to take the following form for $U(2)$ projectors (with the simplest topological charge $k = 1$):

$$
\begin{aligned}
P &= |\psi><\psi| \\
|\psi> &= \left(\begin{Bmatrix} \begin{pmatrix} a_0 & -a_1^\dagger \\ a_1 & a_0^\dagger \end{pmatrix} \\ \begin{pmatrix} a_1 & -a_0^\dagger \\ a_0 & a_1^\dagger \end{pmatrix} \end{Bmatrix} \right) f(\hat{N})
\end{aligned} \tag{3.1}
$$

It is easy to recognize that to obtain a projector different from identity, we need to play with the interference between the first and the second element of the vector $|\psi>$, since

a single component would produce no functional dependence on the elements of the fuzzy sphere. Note the exchange of indices ($0 \leftrightarrow 1$) in the second element of the vectors which is responsible for a non-trivial interference.

We will notice in the following that adding the second element with interchanged indices is also important to find a simple result for the expectation value $< \psi | L_i | \psi >$, the form which has enabled us to derive the non-commutative abelian monopoles in the matrix model formalism.

As in the $U(1)$ case, the function $f(\hat{N})$ can be determined by imposing the normalization condition on $| \psi >$:

$$< \psi | \psi >= f^2(\hat{N}) \begin{pmatrix} 2\hat{N} & 0 \\ 0 & 2(\hat{N}+2) \end{pmatrix} \tag{3.2}$$

Therefore the form of $f(\hat{N})$ is fixed as the following diagonal form :

$$f(\hat{N}) = \begin{pmatrix} \frac{1}{\sqrt{2\hat{N}}} & 0 \\ 0 & \frac{1}{\sqrt{2(\hat{N}+2)}} \end{pmatrix} \tag{3.3}$$

Using the commutation rule of the oscillators, the final form of the vector $| \psi >$ can be simplified as :

$$| \psi >= \frac{1}{\sqrt{2(\hat{N}+1)}} \left(\begin{Bmatrix} a_0 & -a_1^\dagger \\ a_1 & a_0^\dagger \\ a_1 & -a_0^\dagger \\ a_0 & a_1^\dagger \end{Bmatrix} \right) \tag{3.4}$$

The corresponding non-commutative $U(2)$ projector, that will be further elaborated for a possible connection with the 't Hooft-Polyakov monopoles, is given by :

$$P = | \psi >< \psi | = \frac{1}{2(\hat{N}+1)} \left(\begin{matrix} \begin{pmatrix} \hat{N}+1 & 0 \\ 0 & \hat{N}+1 \end{pmatrix} & \begin{pmatrix} 2a_0 a_1^\dagger & a_0 a_0^\dagger - a_1^\dagger a_1 \\ a_1 a_1^\dagger - a_0^\dagger a_0 & 2a_0^\dagger a_1 \end{pmatrix} \\ \begin{pmatrix} 2a_0^\dagger a_1 & a_1 a_1^\dagger - a_0^\dagger a_0 \\ a_0 a_0^\dagger - a_1^\dagger a_1 & 2a_0 a_1^\dagger \end{pmatrix} & \begin{pmatrix} \hat{N}+1 & 0 \\ 0 & \hat{N}+1 \end{pmatrix} \end{matrix} \right) \tag{3.5}$$

At this level, it is safe substituting to the number operator $\hat{N}$ its eigenvalue N, and considering the entries of P as elements of the fuzzy sphere function space.

The trace of the projector is obviously :

$$TrP = 2\,TrI < Tr1_P = 4TrI \tag{3.6}$$

Being this projector non-trivial, it can be taken as a natural candidate for a non-abelian topological configuration. To be sure, we must study its connection with matrix models.

We already know what can be expected from our previous study of $U(1)$ projectors. The combined presence of operators of type $a(a^\dagger)$ implies that this projector will act on the background in order to increase (decrease) the dimension of the representation used in the background. This is exactly the characteristic of a topologically non-trivial configuration that in ref. [25] has been shown to reproduce in the classical limit the 't Hooft-Polyakov monopoles. We feel therefore to be on the right way and the strategy of projectors combined with the use of matrix models can also, as a byproduct, teach us how to treat the non-abelian topology on the classical sphere, generalizing the mathematical work of [8].

4 Connection with matrix models

Tentatively, we can try to connect the projectors with matrix models, as successfully done in the $U(1)$ case,

$$X_i = < \psi | L_i | \psi > \tag{4.1}$$

However we easily recognize that, in the classical limit, this formula is inconsistent with the presence of a non-trivial Higgs field, since the L_i action reduces to $k_i^a \partial_a$ and therefore it projects in the tangent plane to the sphere, while the Higgs field fluctuation is in the orthogonal direction, along the radius. The explicit calculation of the matrix element $< \psi | L_i | \psi >$ will suggest us what we need to add to this formula to complete the connection with matrix models in the non-abelian case.

The explicit calculation gives, in details,

$$< \psi | L_i | \psi > = \frac{1}{2} \begin{pmatrix} \frac{1}{\hat{N}}(a_0^\dagger L_i a_0 + a_1^\dagger L_i a_1) & \frac{1}{\hat{N}}(-a_0^\dagger L_i a_1^\dagger + a_1^\dagger L_i a_0^\dagger) \\ \frac{1}{\hat{N}+2}(-a_1 L_i a_0 + a_0 L_i a_1) & \frac{1}{\hat{N}+2}(a_0 L_i a_0^\dagger + a_1 L_i a_1^\dagger) \end{pmatrix} + (0 \leftrightarrow 1) \tag{4.2}$$

We note from this formula the importance of the contribution ($0 \leftrightarrow 1$) to cancel the off-diagonal terms.

In summary we obtain:

$$< \psi|L_i|\psi >= \begin{pmatrix} \frac{1}{N}(a_0^\dagger L_i a_0 + a_1^\dagger L_i a_1) & 0 \\ 0 & \frac{1}{\hat{N}+2}(a_0 L_i a_0^\dagger + a_1 L_i a_1^\dagger) \end{pmatrix} \qquad (4.3)$$

It is not difficult to compute the terms inside the parenthesis using the oscillator algebra:

$$< \psi|L_i|\psi >= \begin{pmatrix} \frac{\hat{N}-1}{\hat{N}}L_i & 0 \\ 0 & \frac{\hat{N}+3}{\hat{N}+2}L_i \end{pmatrix} \qquad (4.4)$$

As we discussed in our previous article [5], the action of L_i on ψ cannot be smoothly connected to the classical Lie derivative on the sphere unless we project the vector $|\psi>$ on the fuzzy sphere algebra. The only possibility left, with the constraint of keeping invariant the projector P, is dressing $|\psi>$ with a quasi-unitary operator, such that $|\psi'>$ belongs to the fuzzy sphere function space:

$$\begin{aligned} P &= |\psi><\psi| = |\psi'><\psi'| \\ |\psi'> &= |\psi> U \end{aligned} \qquad (4.5)$$

The quasi-unitary operator U, as in the case of non-commutative extension of Dirac monopoles, plays an essential role to define the non-abelian topology. The only consistent choice, apart from unitary gauge transformation, given the structure of the vector $|\psi>$, is the following quasi-unitary operator :

$$\begin{aligned} U &= \begin{pmatrix} U_1^\dagger & U_{12}^\dagger \\ 0 & U_2 \end{pmatrix} \\ U_1 &= \sum_{n_1,n_2=0}^{\infty} |n_1, n_2><n_1+1, n_2| \\ U_2 &= \sum_{n_1,n_2=0}^{\infty} |n_1, n_2><n_1, n_2+1| \\ U_{12} &= \sum_{n_1=0}^{\infty} |n_1, 0><0, n_1+1| \end{aligned} \qquad (4.6)$$

This operator satisfies to the following properties:

$$UU^{\dagger} = \begin{pmatrix} U_1^{\dagger} & U_{12}^{\dagger} \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ U_{12} & U_2^{\dagger} \end{pmatrix} = \begin{pmatrix} U_1^{\dagger}U_1 + U_{12}^{\dagger}U_{12} = 1 - |0><0| & U_{12}^{\dagger}U_2^{\dagger} = 0 \\ U_2 U_{12} = 0 & U_2 U_2^{\dagger} = 1 \end{pmatrix} \tag{4.7}$$

Since the operator $|0><0|$ is annihilated by the action of a_0 and a_1, the combination $UU^{\dagger}$ behaves as the identity when acting on the oscillators.

Moreover the following property holds :

$$U^{\dagger}U = \begin{pmatrix} U_1 & 0 \\ U_{12} & U_2^{\dagger} \end{pmatrix} \begin{pmatrix} U_1^{\dagger} & U_{12}^{\dagger} \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} U_1 U_1^{\dagger} = 1 & U_1 U_{12}^{\dagger} = 0 \\ U_{12}U_1^{\dagger} = 0 & U_2^{\dagger}U_2 + U_{12}U_{12}^{\dagger} = 1 \end{pmatrix} \tag{4.8}$$

This operator doesn't change the rank of the background but simply it changes the dimensions of the component representations. Therefore we must expect that :

$$\begin{pmatrix} (L_i)_{N+1} & 0 \\ 0 & (L_i)_{N+1} \end{pmatrix} \rightarrow \left[U^{\dagger} \begin{pmatrix} L_i & 0 \\ 0 & L_i \end{pmatrix} U \right]_{N+1} = \left(\begin{array}{c|c} (L_i)_{N+2} & 0 \\ \hline 0 & (L_i)_N \end{array} \right) \tag{4.9}$$

where the basic building blocks in the last formula have different size from those of the background.

Now let's apply $|\psi'>$ to the formula(4.1). In this case we obtain

$$\begin{aligned} X_i &= <\psi'|L_i|\psi'> = U^{\dagger} \begin{pmatrix} \frac{\hat{N}-1}{\hat{N}}L_i & 0 \\ 0 & \frac{\hat{N}+3}{\hat{N}+2}L_i \end{pmatrix} U = \\ &= \frac{\hat{N}}{\hat{N}+1} \begin{pmatrix} U_1 L_i U_1^{\dagger} & U_1 L_i U_{12}^{\dagger} \\ U_{12} L_i U_1^{\dagger} & U_{12} L_i U_{12}^{\dagger} \end{pmatrix} + \frac{\hat{N}+2}{\hat{N}+1} \begin{pmatrix} 0 & 0 \\ 0 & U_2^{\dagger} L_i U_2 \end{pmatrix} \end{aligned} \tag{4.10}$$

It is not difficult to recognize that this formula is nothing but a sum of $SU(2)$ representations (see Appendix):

$$\begin{pmatrix} U_1 L_i U_1^{\dagger} & U_1 L_i U_{12}^{\dagger} \\ U_{12} L_i U_1^{\dagger} & U_{12} L_i U_{12}^{\dagger} \end{pmatrix}_{N+1} = \left(\begin{array}{c|c} (L_i)_{N+2} & 0 \\ \hline 0 & 0 \end{array} \right) \tag{4.11}$$

while

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$$\begin{pmatrix} 0 & 0 \\ 0 & U_2^\dagger L_i U_2 \end{pmatrix}_{N+1} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (L_i)_N \end{array} \right) \tag{4.12}$$

In summary, the presence of the quasi-unitary operator U can be completely worked out into this final formula :

$$X_i = <\psi'|L_i|\psi'> = \left(\begin{array}{c|c} \frac{\hat{N}}{\hat{N}+1}(L_i)_{N+2} & 0 \\ \hline 0 & \frac{\hat{N}+2}{\hat{N}+1}(L_i)_N \end{array} \right) \tag{4.13}$$

This block-diagonal form is still not an explicit solution of the equations of motion. The nearest solution is very simple to obtain, redefining $|\psi'>$ with a diagonal matrix :

$$X_i = \begin{pmatrix} f_+ + f_- & 0 \\ 0 & f_+ - f_- \end{pmatrix} < \psi'|L_i|\psi' > \begin{pmatrix} f_+ + f_- & 0 \\ 0 & f_+ - f_- \end{pmatrix} \tag{4.14}$$

This final form is not of type $< \psi'|L_i|\psi' >$, as in the case of Dirac monopoles; however as we discussed in the beginning, this modification is necessary to obtain in the classical limit a non-trivial contribution for the Higgs field.

Finally requiring that

$$X_i = \left(\begin{array}{c|c} (L_i)_{N+2} & 0 \\ \hline 0 & (L_i)_N \end{array} \right) \tag{4.15}$$

we obtain a condition that fixes the unknown constants f_+, f_-:

$$f_\pm = \frac{1}{2} \left[\sqrt{\frac{\hat{N}+1}{\hat{N}}} \pm \sqrt{\frac{\hat{N}+1}{\hat{N}+2}} \right] \tag{4.16}$$

This solution can also be expressed, with a gauge transformation, as:

$$X_i = L_i \otimes 1 + 1 \otimes S_i \tag{4.17}$$

Now it is simpler to extract the contribution of the fluctuation (non-abelian monopole field) from the background:

$$X_i - (\text{background}) = 1 \otimes S_i \tag{4.18}$$

An alternative way to express formula (4.14), reaching the same solution, is redefining both $|\psi'>$ and the projector P as

$$
\begin{aligned}
|\psi''> &= \begin{pmatrix} \frac{1}{f_++f_-} & 0 \\ 0 & \frac{1}{f_+-f_-} \end{pmatrix} |\psi'> \begin{pmatrix} f_++f_- & 0 \\ 0 & f_+-f_- \end{pmatrix} \\
P' &= |\psi''><\psi''| = \begin{pmatrix} \frac{1}{f_++f_-} & 0 \\ 0 & \frac{1}{f_+-f_-} \end{pmatrix} P \begin{pmatrix} f_++f_- & 0 \\ 0 & f_+-f_- \end{pmatrix}
\end{aligned}
\tag{4.19}
$$

The formula, analogous to (4.14), linking projectors to connections, can be expressed in this case as

$$
\begin{aligned}
X_i &= <\psi''| \begin{pmatrix} f_++f_- & 0 \\ 0 & f_+-f_- \end{pmatrix} L_i \begin{pmatrix} f_++f_- & 0 \\ 0 & f_+-f_- \end{pmatrix} |\psi''> \\
&= <\psi''| \begin{pmatrix} \frac{\hat{N}+1}{\hat{N}} L_i & 0 \\ 0 & \frac{\hat{N}+1}{\hat{N}+2} L_i \end{pmatrix} |\psi''> = <\psi''|X_i^0|\psi''>
\end{aligned}
\tag{4.20}
$$

This version allows us to define a gauge invariant version of matrix models, generalizing what we have done in the $U(1)$ case (ref. [5]), built directly on the projectors

$$
X_i = P'X_i^0 P' \tag{4.21}
$$

that, by construction, satisfies the same equations of motion.

5   Classical limit

The lagrangian of the matrix model (eq. (2.14)) for $\lambda = 2$ in the non-abelian case leads to the following classical action on the sphere:

$$
S = -\frac{1}{4g_{YM}^2} \int d\Omega (F_{ab}F_{a'b'}g^{aa'}g^{bb'} + 2g^{aa'}D_a\phi D_{a'}\phi) \tag{5.1}
$$

where we define

$$
\begin{aligned}
D_a\phi &= -i\partial_a\phi + [A_a, \phi] \\
F_{ab} &= -i(\partial_a A_b - \partial_b A_a) + [A_a, A_b]
\end{aligned}
\tag{5.2}
$$

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The variation with respect to A_a and ϕ is vanishing by simply requiring that

$$D_a(\sqrt{g}g^{aa'}g^{bb'}F_{a'b'}) = D_a\phi = 0 \tag{5.3}$$

On the other hand, we know that at non-commutative level

$$X_i = L_i \otimes 1 + 1 \otimes S_i \tag{5.4}$$

satisfies the constraint

$$F_{ij} = [X_i, X_j] - i\epsilon_{ijk}X_k = 0 \tag{5.5}$$

since X_i satisfies the commutation rules of the angular momentum. In the classical limit the fluctuation field A_i

$$A_i = 1 \otimes S_i = k_i^a A_a + n_i\phi \tag{5.6}$$

can be projected over the tangent plane of the sphere and on the orthogonal direction

$$
\begin{aligned}
\phi &= n_i \otimes S_i \\
A_a &= g_{ab}\, k_i^b \otimes S_i
\end{aligned}
\tag{5.7}
$$

We must obtain, as a check, that

$$
\begin{aligned}
F_{ij} &= \frac{1}{R}\epsilon_{ijk}x_k\left(i\phi + \frac{\epsilon^{ab}}{2\sqrt{g}}F_{ab}\right) + \frac{1}{R}x_j k_i^a D_a\phi - \frac{1}{R}x_i k_j^a D_a\phi = 0 \\
D_a\phi &= 0 \\
F_{ab} &= -i\epsilon_{ab}\sqrt{g}\phi
\end{aligned}
\tag{5.8}
$$

We note that these constraints are enough to solve the classical equations of motion (eq. (5.3)) for $\lambda = 2$. This check permits us to have an explicit formula for ϕ and A_a:

$$\phi = n_i \otimes S_i = \frac{1}{2}\begin{pmatrix} cos\theta & sin\theta e^{-i\phi} \\ sin\theta e^{i\phi} & -cos\theta \end{pmatrix}$$

$$A_\theta = k_i^\theta \otimes S_i = \frac{i}{2} \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

$$A_\phi = \sin^2\theta \ k_i^\phi \otimes S_i = \frac{\sin\theta}{2} \begin{pmatrix} \sin\theta & -\cos\theta e^{-i\phi} \\ -\cos\theta e^{i\phi} & -\sin\theta \end{pmatrix} \tag{5.9}$$

It is easy to deduce that

$$F_{\theta\phi} = -\frac{i}{2}(\partial_\theta A_\phi - \partial_\phi A_\theta) = -[A_\theta, A_\phi] = -i\sqrt{g}\phi \tag{5.10}$$

that is equivalent to eq. (5.8), with the notation $\epsilon_{\theta\phi} = 1$.

Moreover

$$-i\partial_\theta\phi = -[A_\theta, \phi] = -\frac{i}{2} \begin{pmatrix} -\sin\theta & \cos\theta e^{-i\phi} \\ \cos\theta e^{i\phi} & \sin\theta \end{pmatrix}$$

$$-i\partial_\phi\phi = -[A_\phi, \phi] = \frac{\sin\theta}{2} \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \tag{5.11}$$

The classical limit induced by our formula (4.14) produces a slightly different solution. Starting from the classical limit of the vector $|\psi'>$:

$$|\psi'> \to_{N\to\infty} |\psi_{cl}> = \frac{1}{\sqrt{2}} \left(\left\{ \begin{pmatrix} a_0 & -a_1^* \\ a_1 & a_0^* \end{pmatrix} \\ \begin{pmatrix} a_1 & -a_0^* \\ a_0 & a_1^* \end{pmatrix} \right\} \right) \qquad <\psi_{cl}|\psi_{cl}> = 1 \tag{5.12}$$

we obtain that the variable X_i of the classical matrix model is of the type:

$$X_i \simeq (f_+^2 + f_-^2) <\psi_{cl}|L_i|\psi_{cl}> +2f_+f_- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} <\psi_{cl}|L_i|\psi_{cl}> \tag{5.13}$$

Since

$$f_+ \to 1 \quad f_- \to \frac{1}{2N} \tag{5.14}$$

we can deduce that

$$X_i \simeq < \psi_{cl}|L_i|\psi_{cl} > + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} < \psi_{cl}|\frac{\hat{x}_i}{R}|\psi_{cl} > \qquad (5.15)$$

where $\hat{x}_i = \alpha L_i$ is the operator deforming the coordinate x_i. The Higgs field contribution comes exactly from this last term:

$$\begin{aligned}
\phi &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
A_\theta &= -i < \psi_{cl}|\partial_\theta|\psi_{cl} > = 0 \\
A_\phi &= -i < \psi_{cl}|\partial_\phi|\psi_{cl} > = -\frac{cos\theta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \qquad (5.16)$$

with the choice

$$\begin{aligned}
a_0 &= cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\
a_1 &= sin\frac{\theta}{2}e^{i\frac{\phi}{2}}
\end{aligned} \qquad (5.17)$$

Naturally there exists a classical gauge transformation that connects the solution (5.9) to the solution (5.16)

$$\begin{aligned}
\phi_g &= n_i \otimes S_i & \phi &= S_3 \\
A_{ag} &= g_{ab}k_i^b \otimes S_i & A_\theta &= 0 \quad A_\phi = -cos\theta S_3
\end{aligned} \qquad (5.18)$$

In both cases the constraint $F_{ij} = 0$ is satisfied. Starting for example from the Higgs field :

$$\phi_g = g^{-1}\phi g = \frac{1}{2} \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \qquad a\overline{a} + b\overline{b} = 1 \qquad (5.19)$$

the gauge transformation is defined by :

$$\begin{aligned}
a\overline{a} - b\overline{b} &= cos\theta \\
2\overline{a}b &= sin\theta e^{-i\phi} \\
a\overline{a} + b\overline{b} &= 1
\end{aligned} \qquad (5.20)$$

which is solved by

$$
\begin{aligned}
a &= cos\frac{\theta}{2}e^{i\frac{\phi}{2}} \\
b &= sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}
\end{aligned}
\tag{5.21}
$$

We have checked that A_{ag} and A_a are related by the same gauge transformation.

6 Deforming the Chern Class

In literature a candidate for an eventual non-commutative topological index has been proposed, by using an action taking values only in the integer numbers [25]. In this paper we want to suggest an alternative, more traditional, definition, by taking an action taking values not necessarily in the integers.

Our candidate for the non-commutative Chern class is the Chern-Simons term

$$
S_{CS} = \frac{1}{N+1}Tr\left[\frac{2i}{3}\epsilon_{ijk}X_iX_jX_k + X_iX_i\right]
\tag{6.1}
$$

This action, as we have found in a previous paper (ref. [26]), is invariant under deformed diffeomorphisms, a property which doesn't hold for the standard Yang-Mills action.

We want to show that this action in the classical limit, once evaluated on the 't Hooft-Polyakov monopole, corresponds to the classical topological number

$$
Q = -\frac{1}{8\pi}\int_S d\Omega\epsilon_{ijk}n_i\epsilon^{abc}n^a(\partial_j n^b)(\partial_k n^c) = -\frac{1}{4\pi}\int_S d\Omega = -1
\tag{6.2}
$$

Comparing with ref. [25], the integral Q is equivalent to the following action

$$
Q = \frac{1}{8\pi}Tr\int_S d\Omega\epsilon_{ijk}n_iF_{jk}^{\perp}\phi
\tag{6.3}
$$

where

$$
\begin{aligned}
\phi &= n_i \otimes S_i \\
F_{ij}^{\perp} &= \partial_i a_j^{\perp} - \partial_j a_i^{\perp} - i[a_i^{\perp}, a_j^{\perp}]
\end{aligned}
\tag{6.4}
$$

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and $a_i^\perp$ is the orthogonal part of the fluctuation A_i, i.e. in our notations

$$a_i^\perp = \epsilon_{ijk}n_j A_k \tag{6.5}$$

Extracting the contribution of the transverse part and substituting the explicit relation between F_{ab} and ϕ given by (5.10)

$$\epsilon_{ijk}n_i F_{ij}^\perp = -2\phi = -i\left(\frac{\epsilon^{ab}}{\sqrt{g}}F_{ab}\right) \tag{6.6}$$

we can prove that

$$Q = \frac{1}{8\pi}Tr\int_S d\Omega \epsilon_{ijk}n_i\ F_{jk}^\perp\phi = -\frac{1}{8\pi}Tr\int_S d\Omega\left[i\phi\frac{\epsilon^{ab}}{\sqrt{g}}F_{ab}\right] \tag{6.7}$$

This expression is very similar to the classical limit of the Chern-Simons topological term (6.1). In fact, from references [12] and [14] we obtain that

$$S_{CS} \to_{\alpha\to 0} \frac{1}{8\pi}Tr\int d\Omega\left(i\phi\frac{\epsilon^{ab}}{\sqrt{g}}F_{ab} - \phi^2\right) \tag{6.8}$$

Applying again the classical limit of the constraint $F_{ij} = 0$ i.e.

$$\begin{aligned}
F_{ab} &= -i\epsilon_{ab}\sqrt{g}\phi \\
S &\to_{\alpha\to 0} \frac{1}{8\pi}Tr\int d\Omega\left(i\phi\frac{\epsilon^{ab}}{2\sqrt{g}}F_{ab}\right) = -\frac{Q}{2} = \frac{1}{2}
\end{aligned} \tag{6.9}$$

and substituting the explicit value for ϕ given by eq. (5.9) we obtain perfect agreement.

We have proved that, at least in the classical limit, the Chern-Simons action, evaluated on the classical solution (5.6), produces the topological number characteristic of non-abelian monopoles.

Now we want to show that the same topological number can be obtained evaluating the Chern-Simons action at pure non-commutative level:

$$S_{CS}^{\mathrm{monopole}} - S_{CS}^{\mathrm{bg}} = -\frac{Q}{2} = \frac{1}{2} \tag{6.10}$$

and that in the non-abelian case there is a continuity with the non-commutative extension. If we evaluate S_{CS} on the background we obtain :

$$S_{CS}^{\text{bg}} = \frac{1}{3(N+1)} Tr \begin{pmatrix} (L_i L_i)_{N+1} & 0 \\ 0 & (L_i L_i)_{N+1} \end{pmatrix} = \frac{N(N+2)}{6} \tag{6.11}$$

while on the monopole configuration

$$S_{CS}^{\text{monopole}} = \frac{1}{3(N+1)} Tr \left(\begin{array}{c|c} (L_i L_i)_{N+2} & 0 \\ \hline 0 & (L_i L_i)_N \end{array} \right) = \frac{N^2 + 2N + 3}{6} \tag{6.12}$$

Therefore

$$S_{CS}^{\text{monopole}} - S_{CS}^{\text{bg}} = \frac{1}{8\pi} \int_S Tr\phi^2 = \frac{1}{2} \tag{6.13}$$

the classical topological number is maintained by the non-commutative deformation in the non-abelian case.

To complete the picture, we want to analyze what happens for Dirac monopoles. At first sight, the classical limit of the Chern-Simons term doesn't seem to be in agreement with the Chern class

$$S_{Ch} = \int_S d\Omega \left(i\frac{\epsilon^{ab}}{\sqrt{g}} F_{ab} \right) = Q \tag{6.14}$$

because of the absence of the Higgs field. This problem can be easily circumvented by admitting that the classical monopole configurations have also a constant Higgs field. Since the lagrangian of the Higgs field is decoupled, this configuration is still solution of the equations of motion.

Therefore the evaluation of S_{CS} is equivalent to the evaluation of S_{Ch}, by allowing the presence of a constant Higgs field. This different classical limit can be realized starting from

$$X_i^{NC} = U^\dagger L_i U \tag{6.15}$$

instead of

$$X_i^{NC} = \frac{N+2}{N+1} U^\dagger L_i U \tag{6.16}$$

This new solution satifies the same commutation rules of the angular momentum and therefore to the constraint

$$F_{\theta\phi} = -i\partial_\theta A_\phi = i\epsilon_{\theta\phi}\sqrt{g}\phi \leftarrow_{\alpha\to 0} \quad F_{ij} = 0 \tag{6.17}$$

and the explicit solution is

$$X_i^{NC} \to_{\alpha\to 0} \begin{cases} A_\phi = \frac{cos\theta-1}{2} \\ \phi = -\frac{1}{2} \end{cases} \tag{6.18}$$

In summary the classical fluctuation that must be compared with the solution X_i^{NC} is of the type:

$$
\begin{aligned}
a_i &= k_i^a A_a + \frac{x^i}{R}\phi \\
k_i^a A_a &= k_i^\phi A_\phi = -\frac{1}{2}\left[\frac{cos\theta}{sin\theta}(cos\theta-1)cos\phi, \frac{cos\theta}{sin\theta}(cos\theta-1)sin\phi, 1-cos\theta\right] \\
n^i\phi &= -\frac{1}{2}[sin\theta cos\phi, sin\theta sin\phi, cos\theta] \\
a_i &\sim -\frac{1}{2}\left[\frac{1-cos\theta}{sin\theta}cos\phi, \frac{1-cos\theta}{sin\theta}sin\phi, 1\right] \simeq (L_i)_N - (L_i)_{N+1}
\end{aligned} \tag{6.19}
$$

The direct non-commutative evaluation in this case produces a singularity:

$$
\begin{aligned}
S_{CS}^{bg} &= \frac{1}{3(N+1)}Tr(L_iL_i)_{N+1} = \frac{N(N+2)}{12} \\
S_{CS}^{monopole} &= \frac{1}{3(N+1)}Tr\begin{pmatrix} (L_iL_i)_N & 0 \\ 0 & 0 \end{pmatrix} = \frac{N(N-1)}{12} \\
S_{CS}^{monopole} - S_{CS}^{bg} &= \frac{N(N-1)}{12} - \frac{N(N+2)}{12} = -\frac{N}{4}
\end{aligned} \tag{6.20}
$$

This negative result requires to analyze more carefully the classical limit of the Dirac monopoles. Starting from the components $a_\pm$

$$
\begin{aligned}
a_+ &= (U^\dagger L_+ U)_{N+1} - (L_+)_{N+1} = \sum(\sqrt{n_1} - \sqrt{n_1+1})\sqrt{n_2+1}|n_1+1, n_2 >< n_1, n_2+1| \\
&\simeq -\frac{1}{2}\sum\sqrt{\frac{n_2+1}{n_1+1}}|n_1+1, n_2 >< n_1, n_2+1| = -\frac{1}{2}\frac{1}{\hat{z}_1}\hat{z}_2
\end{aligned} \tag{6.21}
$$

where we have defined

$$\frac{1}{\hat{z}_1} = \sum_{n_1} \frac{1}{\sqrt{n_1+1}} |n_1+1><n_1| \qquad z_1 = \sqrt{2}\,\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}$$

$$\hat{z}_2 = \sum_{n_2} \sqrt{n_2+1}|n_2><n_2+1| \qquad z_2 = \sqrt{2}\,\sin\frac{\theta}{2}e^{i\frac{\phi}{2}}$$

$$x_+ = \bar{z}_1 z_2 = \sin\theta e^{i\phi} \qquad x_3 = \cos\theta$$

$$a_+ \rightarrow -\frac{1}{2z_1}z_2 = -\frac{1}{2}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}e^{i\phi} = \frac{1}{2}\frac{\cos\theta-1}{\sin\theta}e^{i\phi} \tag{6.22}$$

we find that a_+ and a_- are really continuous deformations of the classical configurations. Different is the case of a_3:

$$a_3 = (U^\dagger L_3 U)_{N+1} - (L_3)_{N+1} = \frac{1}{2}\left(-\sum_{n_1\neq 0}|n_1,n_2><n_1,n_2| + \sum_{n_2} n_2|0,n_2><0,n_2|\right)_{N+1} =$$

$$= -\left(\frac{1-P_0}{2}\right) + \frac{N+1}{2}P_0 \qquad P_0 = |0,N+1><0,N+1| \tag{6.23}$$

a_3 has a problem, i.e. a discontinuous term that has support on a single state:

$$a_3^m = -\left(\frac{1-P_0}{2}\right)$$

$$a_3^s = \frac{N+1}{2}P_0 \tag{6.24}$$

The effect of a_3^s seems at first sight negligible, if we compare a single state with respect to an infinite number. However on the non-commutative action it makes a difference and produces a nasty discontinuity with the classical topological number. Let us try to exclude the contribution of this state, redefining the solution as

$$X_i = (L_i + a_i^s) + a_i^m$$

$$X_i' = X_i - a_i^s = L_i + a_i^m$$

$$S_{CS}^{\text{monopole}} - S_{CS}^{\text{bg}} = S_{X'} - S_{L_i} = \frac{N^2+2N+3}{12} - \frac{N(N+2)}{12} = \frac{1}{4} =$$

$$= \frac{1}{4\pi}\int_S d\Omega\left(i\phi\frac{\epsilon^{ab}}{2\sqrt{g}}F_{ab}\right) = -\frac{Q}{4} = \frac{1}{4} \tag{6.25}$$

The agreement now is perfect.

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7   Conclusions

In this work we have shown how to characterize easily the topologically nontrivial configurations leading to 't Hooft-Polyakov monopoles with a noncommutative $U(2)$ projector. The non-triviality of the $U(2)$ projector is assured in the classical limit by the use of the Hopf fibration $\pi : S^3 \to S^2$, since the projector cannot be decomposed in terms of vectors belonging to the sphere S^2 but only to S^3.

At noncommutative level we can surely state that there exist solutions to the matrix model equations of motion extending in a smooth way classical topology.

It remains an open question how to characterize the topological meaning of these non-commutative configurations, whether they remain stable and so on.

As a first step in this direction we have suggested a candidate for deforming the Chern class, maintaining invariant the classical topological number. However, this subject still requires a deeper investigation in the future.

A   Appendix

Aim of this appendix is showing that the action of the quasi-unitary operator produces the following modification of the representations of the Lie algebra:

$$(U^\dagger L_i U)_{N+1} = \left[\begin{pmatrix} U_1 & 0 \\ U_{12} & U_2^\dagger \end{pmatrix} L_i \begin{pmatrix} U_1^\dagger & U_{12}^\dagger \\ 0 & U_2 \end{pmatrix} \right]_{N+1} = \left(\begin{array}{c|c} (L_i)_{N+2} & 0 \\ \hline 0 & (L_i)_N \end{array} \right) \qquad \text{(A.1)}$$

where

$$\begin{aligned}
U_1 &= \sum_{n_1,n_2} |n_1, n_2 >< n_1 + 1, n_2| \\
U_2 &= \sum_{n_1,n_2} |n_1, n_2 >< n_1, n_2 + 1| \\
U_{12} &= \sum_{n_2} |n_1, 0 >< 0, n_1 + 1| \qquad \text{(A.2)}
\end{aligned}$$

We firstly observe that the following part is trivially verified

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & U_2^\dagger \end{pmatrix} L_i \begin{pmatrix} 0 & 0 \\ 0 & U_2 \end{pmatrix} \right]_{N+1} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (L_i)_N \end{array} \right) \qquad \text{(A.3)}$$

and that the component U_2 is practically decoupled from the others.

We must show that

$$\left[\begin{pmatrix} U_1 & 0 \\ U_{12} & 0 \end{pmatrix} L_i \begin{pmatrix} U_1^\dagger & U_{12}^\dagger \\ 0 & 0 \end{pmatrix}\right]_{N+1} = \left(\begin{array}{c|c} (L_i)_{N+2} & 0 \\ \hline 0 & 0 \end{array}\right) \tag{A.4}$$

We start considering the diagonal matrix L_3 :

$$L_3 = \sum_{n_1,n_2} (n_1 - n_2)|n_1, n_2 >< n_1, n_2| \tag{A.5}$$

The term

$$U_1 L_3 U_1^\dagger = \sum_{n_1,n_2} (n_1 + 1 - n_2)|n_1, n_2 >< n_1, n_2| \tag{A.6}$$

once that it is restricted to the component with total number $n_1 + n_2 = N + 1$ gives rise to

$$(U_1 L_3 U_1^\dagger)_{N+1} = \sum_{k=0}^{N+1} (N + 2 - 2k)|N + 1 - k, k >< N + 1 - k, k| \tag{A.7}$$

It is a diagonal matrix, having as entries $(N+2), ..., -N$. To complete the representation of $(L_3)_{N+2}$, it is enough to add a matrix element with value $-(N + 2)$.

We note that, by construction, the off-diagonal elements are null

$$U_1 L_3 U_{12}^\dagger = U_{12} L_3 U_1^\dagger = 0 \tag{A.8}$$

and therefore the remaining matrix element must arise from the term

$$U_{12} L_3 U_{12}^\dagger = - \sum_{n_1=n_2} (n_2 + 1)|n_1, 0 >< n_1, 0| \tag{A.9}$$

When projecting the operator to the component $n_1 = n_2 = (N + 1)$, we obtain :

$$(U_{12} L_3 U_{12}^\dagger)_{N+1} = -(N + 2)|N + 1, 0 >< N + 1, 0| \tag{A.10}$$

that is exactly the right term in the right position to complete the representation

$$\left(\begin{array}{c|c} (L_3)_{N+2} & 0 \\ \hline 0 & 0 \end{array} \right) \qquad (A.11)$$

Now we are going to consider the case of L_+

$$L_+ = a_0^\dagger a_1 = \sum_{n_1, n_2} \sqrt{(n_1 + 1)(n_2 + 1)} |n_1 + 1, n_2 >< n_1, n_2 + 1| \qquad (A.12)$$

By construction we obtain

$$\begin{aligned} U_1 L_+ U_1^\dagger &= \sum_{n_1, n_2} \sqrt{(n_1 + 2)(n_2 + 1)} |n_1 + 1, n_2 >< n_1, n_2 + 1| \\ (U_1 L_+ U_1^\dagger)_{N+1} &= \sum_{k=0}^{N} \sqrt{(N - k + 2)(k + 1)} |N - k + 1, k >< N - k, k + 1| \quad (A.13) \end{aligned}$$

This term is part of the representation $(L_+)_{N+2}$, without the term $k = N + 1$.

By looking at the explicit matrix L_+, we expect that the term $k = N + 1$ must appear in the combination $U_1 L_+ U_{12}^\dagger$. in fact it is easy to show that

$$U_{12} L_+ U_1^\dagger = U_{12} L_+ U_{12}^\dagger = 0 \qquad (A.14)$$

The explicit calculation gives for $U_1 L_+ U_{12}^\dagger$

$$U_1 L_+ U_{12}^\dagger = \sum_{n_1 = n_2} \sqrt{n_2 + 1} |0, n_2 >< n_1, 0| \qquad (A.15)$$

and its projection to the term $n_1 = n_2 = N + 1$ implies that

$$(U_1 L_+ U_{12}^\dagger)_{N+1} = \sqrt{N + 2} |0, N + 1 >< N + 1, 0| \qquad (A.16)$$

that is what we need to complete the representation $(L_+)_{N+2}$, Analogous proof holds for L_-.

References

[1] S. Baez, A.P. Balachandran, B. Ydri, S. Vaidya, " Monopoles and Solitons in fuzzy physics ", Comm. Math. Phys. **208** (2000) 787,hep-th/9811169.

[2] A.P. Balachandran and S. Vaidya, " Instantons and chiral anomaly in fuzzy physics ", hep-th/9910129.

[3] A. P. Balachandran, X.Martin and D. O' Connor, " Fuzzy actions and their continuum limits ", hep-th/0007030.

[4] J. Madore, " The Fuzzy sphere ", Class. Quantum Grav. **9** (1992) 69.

[5] P. Valtancoli, " Projectors, matrix models and noncommutative monopoles ",Int. J. Mod. Phys. **A19** (2004) 4641 , hep-th/0404045.

[6] M.Dubois-Violette, Y. Georgelin, " Gauge theory in terms of projector valued fields ", Phys. Lett. B **82** 251, 1979.

[7] T. Eguchi, P. B. Gilkey, A. J. Hanson, " Gravitation, gauge theories and differential geometry ", Phys.Rept. **66** 213,1980.

[8] G. Landi, " Projective modules of finite type and monopoles over S^2 " , J.Geom.Phys. **37** 47, 2001, math-ph/9905014.

[9] P. Valtancoli, " Projectors for the fuzzy sphere " Mod.Phys.Lett. **A16** (2001) 639, hep-th/0101189.

[10] P. Valtancoli, " Stability of the fuzzy sphere solution from matrix model", Int. J. Mod. Phys. **A18** (2003), 967; hep-th/0206075

[11] P. Valtancoli, " Solitons for the fuzzy sphere from matrix model", Int. J. Mod. Phys. **A18** (2003), 1107; hep-th/0209117.

[12] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, " Noncommutative gauge theory on fuzzy sphere from Matrix model ", Nucl. Phys. **B604** (2001) 121, hep-th/0101102.

[13] R.C. Myers, "Dielectric branes", JHEP **9912** (1999) 022, hep-th/9910053.

[14] Y. Kimura, " Noncommutative gauge theories on fuzzy sphere and fuzzy torus from Matrix model ", Prog. Theor. Phys. **106** (2001) 445, hep-th/0103192.

[15] D. Berenstein, J. M. Maldacena, H. Nastase, " Strings in flat space and PP waves from N=4 Superyang-mills " , JHEP **0204** (2002) 013, hep-th/0202021.

[16] J. Ambjorn, Y.M. Makeenko, J. Nishimura and R. J. Szabo, " Finite N Matrix models of noncommutative gauge theory ", JHEP **9911** (1999) 029, hep-th/9911041.

[17] J. Ambjorn, Y.M. Makeenko, J. Nishimura and R. J. Szabo, " Non perturbative Dynamics of noncommutative gauge theory ", Phys. Lett. **B480** (2000) 399.

[18] J. Ambjorn, Y.M. Makeenko, J. Nishimura and R. J. Szabo, " Lattice gauge fields and discrete noncommutative Yang-Mills Theory ", JHEP **0005** (2000) 023, hep-th/0004147.

[19] H. Aoki, S. Iso, H, Hawai, Y. Kitazawa, T. Tada and A. Tsuchiya, " IIB matrix model ", Prog. Theor. Phys. Suppl. **134** (1999) 47, hep-th/9908038.

[20] H. Aoki, S. Iso, H. Hawai, Y. Kitazawa and T. Tada, " Space-time structures from IIB Matrix model ", Prog. Theor. Phys. **99** (1998) 713, hep-th/9802085.

[21] M. Li, " Strings from IIB matrices ", Nucl. Phys. **B499**(1997) 149, hep-th/9612222.

[22] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, " Noncommutative Tang-Mills in IIB Matrix model ", Nucl. Phys. **B565** (2000) 176, hep-th/9908141.

[23] P. Valtancoli, " Projective modules over the fuzzy four sphere", Mod. Phys. Lett. **A17** (2002) 2189; hep-th/0210166.

[24] P. Valtancoli, " Projectors, matrix models and noncommutative instantons ", Int. J. Mod. Phys. **A19** (2004) 4789, hep-th/0404046.

[25] H. Aoki, S. Iso and K. Nagao, " Ginsparg-Wilson relation and 't Hooft-Polyakov Monopole on Fuzzy 2-Sphere ", Nucl. Phys. **B684** (2004) 162, hep-th/0312199.

[26] P. Valtancoli, " Gravity on a fuzzy sphere ", Int. J. Mod. Phys. **A19** (2004) 361, hep-th/0306065.